Classification theory for anequilibrium phase transitions

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The paper introduces a classification of phase transitions in which each transition is characterized through its generalized order and a slowly varying function. This characterization is shown to be applicable in statistical mechanics as well as in thermodynamics albeit for different mathematical reasons. By introducing the block ensemble limit the statistical classification is based on the theory of stable laws from probability theory. The block ensemble limit combines scaling limit and thermodynamic limit. The thermodynamic classification on the other hand is based on generalizing Ehrenfest's traditional classification scheme. Both schemes imply the validity of scaling at phase transitions without the need to invoke renormalizaton-group arguments. The statistical classification scheme allows derivation of a form of finite-size scaling for the distributions of statistical averages while the thermodynamic classification gives rise to multiscaling of thermodynamic potentials. The different nature of the two classification theories is also apparent from the fact that the generalized thermodynamic order is unbounded while the statistical order is restricted to values less than 2. This fact is found to be related to the breakdown of hyperscaling relations. Both classification theories predict the possible existence of phase transitions having orders less than unity. Such transitions are termed anequilibrium transitions. Systems near anequilibrium transitions cannot be described by conventional equilibrium thermodynamics or equilibrium statistical mechanics because of very strong fluctuations. Anequilibrium transitions are found to exist in statistical-mechanical model systems. The identification of the Lagrange parameter β in the canonical ensemble becomes invalid if a reservoir and a system of the same substance are in thermal contact and anequilibrium transitions are present. Based on the ergodic hypothesis and the theory of convolution semigroups it is shown that near anequilibrium transitions the equations of motion for macroscopic observables of infinite systems may involve modified time derivatives as generators of the macroscopic time evolution. The general solution to the modified equations of motion exhibits very slow dynamics as frequently observed in a nonequilibrium experiment.

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I. INTRODUCTION

A number of recent publications [1–5] has reopened the discussion concerning the classification of phase transitions. In [1] a thermodynamic classification of normal and anomalous first-order phase transitions was given. In [3] and [4] it was shown that also continuous phase transitions can be usefully classified by extending the thermodynamic classification scheme of Ehrenfest. Based on the generalized classification scheme a class of phase transitions having order less than unity was conjectured to exist [5] and it was shown that this transition type is allowed by the laws of classical thermodynamics. However, the identification of the corresponding statistical-mechanical classification theory remained incomplete. Transitions of order less than 1 will be called *anequilibrium transitions* in this paper.

My objective in this paper is to discuss in more detail the statistical-mechanical (SM) and the thermodynamical (TD) classification theories. To this end the classification scheme introduced in [3] and [4] will first be refined. Phase transitions of order less than unity [5] will be discussed thermodynamically. Then the statistical-mechanical classification is shown to be related to finite-size-scaling theory and the breakdown of hyperscaling other than through the traditional mechanism [6]. Next it is shown that canonical descriptions of systems which are a subregion of an infinite sample may require a renormalization of temperature. Finally some general consequences for the dynamics of critical systems will be discussed.

Discontinuities and divergences of thermodynamic potentials along curves which cross a critical manifold can be characterized mathematically through their generalized orders $\lambda \geq 1$ (with $\lambda \in \mathbb{R}$) as well as their strengths [3,4]. Besides providing a convenient language the classification scheme implies multiscaling for thermodynamic phase transitions [3,4]. Multiscaling has been defined as a scaling form in which the critical exponents are functions of the scaling variables [7]. The simplest form of multiscaling occurs at a multicritical point but recently more interesting cases have been discussed [8–16]. The derivation of multiscaling from analytic continuation of the Ehrenfest scheme represents a derivation of thermodynamic scaling results without the use of renormalization-group ideas.

Given these results it was natural to ask whether phase transitions of order $\lambda < 1$ are thermodynamically admissible or not. A rash answer would be negative because such transitions appear to violate thermodynamic stabili-

ty requirements. A more cautious response, however, is useful. It was shown in [5] that transitions having $\lambda<1$ are allowed by the laws of thermodynamics. Here the consequences of this discovery for statistical mechanics will be studied in more detail while more general consequences have been discussed elsewhere [17].

II. THERMODYNAMICS

A. Refined thermodynamic classification scheme

Let me begin by recalling the definition of the generalized order of a transition [3,4] as well as some of the mathematical requirements of thermodynamics [18–22]. The energy function $U(S,V,N)$ must be a single-valued, convex, monotonically increasing, and almost everywhere differentiable function which is homogeneous of degree 1 and has the coordinates S, entropy, V, volume, and N particle number. Classically the state variables satisfy $0\le V<\infty$, $0\le N<\infty$, $-\infty<S<\infty$, and $-\infty<U<\infty$, while for quantum systems S and U must also be bounded from below. These conditions are both necessary and sufficient for thermodynamic stability.

The classification of phase transitions is usually discussed in terms of the free-energy density or the pressure p [1,3,4] because other thermodynamic potentials are continuous for large classes of interactions [1,2,23]. The pressure is the *conjugate convex function* [24] to the energy density $u(s,\rho)=U(S/V,1,N/V)/V$ as a function of entropy density $s=S/V$ and particle-number density $\rho=N/V$ according to

$$p(T,\mu)=\sup_{s,\rho}[\mu\rho+Ts-u(s,\rho)]\ , \tag{2.1}$$

where μ denotes the chemical potential and T the temperature. The existence of phase transitions requires the thermodynamic limit $V\to\infty$ to be taken appropriately. Consider a thermodynamic process $\mathcal{C}:\mathbb{R}\to\mathbb{R}^2$, $\sigma\mapsto(T(\sigma),\mu(\sigma))$ parametrized by σ such that $T(\sigma=0)=T_c$, $\mu(\sigma=0)=\mu_c$ corresponds to a critical point. The classification scheme for phase transitions [3,4] is based on the fractional derivatives [25]

$$\begin{aligned}\mathcal{F}(\mathcal{C},q;\sigma)&=\frac{d^q p_{\rm sng}(T(\sigma),\mu(\sigma))}{d\sigma^q}\\&=\lim_{N\to\infty}\Gamma(-q)^{-1}\left[\frac{|\sigma|}{N}\right]^{-q}\sum_{j=0}^{N-1}\frac{\Gamma(j-q)}{\Gamma(j+1)}p_{\rm sng}\left[T\left[\sigma-\frac{j\sigma}{N}\right],\mu\left[\sigma-\frac{j\sigma}{N}\right]\right],\end{aligned} \tag{2.2}$$

when $p_{\rm sng}$ denotes the singular part of the pressure $p=p_{\rm reg}+p_{\rm sng}$ and $p_{\rm reg}$ is the regular part. In Ehrenfest's original classification scheme [26] a phase transition was defined to be of order $n\in\mathbb{N}$ iff

$$\mathcal{F}(\mathcal{C},n;\sigma)\sim A\Theta(\sigma)+B \tag{2.3}$$

for $\sigma\approx 0$ where $A,B\in\mathbb{R}$ and $\Theta(\sigma)$ denotes the Heaviside step function defined as $\Theta(\sigma)=1$ for $\sigma>0$ and $\Theta(\sigma)=0$ for $\sigma<0$. Equation (2.3) expresses a finite jump discontinuity in the nth-order derivative of the pressure.

In [3] and [4] the classification scheme of Ehrenfest was generalized by extending the order n from integers to real numbers. A phase transition was defined to be of order $\lambda^{\pm}\in\mathbb{R}$ iff

$$\lambda^{\pm}(\mathcal{C})=\sup\{q\in\mathbb{R}\,|\lim_{\sigma\to 0\pm}\mathcal{F}(\mathcal{C},q;\sigma)<\infty\} \tag{2.4}$$

which is sufficiently general to allow confluent logarithmic singularities. Note that the order will in general depend upon the particular choice of thermodynamic process $\mathcal{C}$. A phase transition of order λ implies that f behaves asymptotically like a power function (of index λ) upon approach to the critical point [3,4]. This observation relates the order of the transition to the critical exponents as

$$\lambda_{\mathcal{E}}=2-\alpha_{\mathcal{E}}=2-\alpha\ , \tag{2.5a}$$

$$\lambda_{\Psi}=2-\alpha_{\Psi}=1+1/\delta\ , \tag{2.5b}$$

where $\lambda_{\mathcal{E}}$ denotes the thermal order and λ_{Ψ} the order along the direction of the field conjugate to the order parameter. In Eq. (2.5) $\alpha_{\mathcal{E}}=\alpha$ and $\alpha_{\Psi}=1-1/\delta$ are the thermodynamic fluctuation exponents for the energy density $\mathcal{E}$ and the order-parameter density Ψ in Fisher's notation [6,27], where δ denotes the equation-of-state exponent and α the specific-heat exponent.

In this paper a mathematically more refined classification scheme will be introduced based on the observation that the function $\mathcal{F}(\mathcal{C},n;\sigma)$ in Ehrenfest's scheme (2.3) is a slowly varying function [28] of σ. A function $\Lambda(x)$ is called *slowly varying at infinity* if it is real valued, positive, and measurable on $[A,\infty)$ for some $A>0$, and if

$$\lim_{x\to\infty}\frac{\Lambda(bx)}{\Lambda(x)}=1 \tag{2.6}$$

for all $b>0$. A function $\Lambda(x)$ is called *slowly varying at zero* if $\Lambda(1/x)$ is slowly varying at infinity [28,29]. The function $\mathcal{F}(\mathcal{C},n;\sigma)$ in (2.2) is slowly varying for $\sigma\to 0+$ as well as for $\sigma\to 0-$. Therefore in this paper *a phase transition is defined to be of order* $\lambda^{\pm}$ iff

$$\lim_{\sigma\to\pm\infty}\frac{\mathcal{F}(\mathcal{C},\lambda^{\pm};b/\sigma)}{\mathcal{F}(\mathcal{C},\lambda^{\pm};1/\sigma)}=1 \tag{2.7}$$

for all $b>0$. This means that $\mathcal{F}(\mathcal{C},\lambda^{\pm};\sigma)$ varies slowly as a function of σ for $\sigma\to 0\pm$. The generalized order in this *refined classification scheme* is the same as in the scheme (2.4) because every slowly varying function $\Lambda(x)$ has the property that $\lim_{x\to 0}x^{-\epsilon}\Lambda(x)=\infty$ and $\lim_{x\to 0}x^{\epsilon}\Lambda(x)=0$ for all $\epsilon>0$.

In the refined classification scheme (2.7) each phase transition is classified by its generalized left and right or-

ders $\lambda^{\pm}$ and functions $\Lambda^{\pm}$ which are slowly varying at the critical point. The classification scheme also allows to distinguish differences between transitions having the same order. The two-dimensional Ising model is of second order $(\lambda,\Lambda)=(2,\log)$ while the mean-field theory will be classed as second order $(\lambda,\Lambda)=(2,\Theta)$ where Θ denotes the Heaviside step function defined above.

Phase transitions of order $\lambda=2$ occupy a special place in the thermodynamic classification scheme because they are self-conjugate under Legendre transformation as will be shown next. Consider a thermal phase transition of order (λ,Λ) for $\tau\to 0^+$ in $p(\tau,\mu)$ where $\tau=(T-T_c)/T_c$ and $\mu=\mu_0$ is constant. Then $p(\tau)$ behaves as

$$p(\tau)\sim\tau^{\lambda}\Lambda(\tau) \tag{2.8}$$

for $T\to T_c^+$ where $\Lambda(\tau)$ is slowly varying for $\tau\to 0^+$. Define a slowly varying function $L(x)$ through

$$\Lambda(x)=\frac{1}{\lambda}L^{(\lambda-1)/\lambda}(x^{\lambda}) . \tag{2.9}$$

It is a standard result in the theory of slowly varying functions that for $\lambda>1$ the conjugate convex function $u(\sigma)=\sup_{\tau}[\tau\sigma-p(\tau)]$ behaves as

$$u(\sigma)\sim\frac{1}{\lambda^*}\sigma^{\lambda^*}L^{*(\lambda^*-1)/\lambda^*}(\sigma^{\lambda^*}) \tag{2.10}$$

for $\sigma\to 0^+$ where $\lambda^*>1$ is given by

$$\lambda^*=\frac{\lambda}{\lambda-1} \tag{2.11}$$

and $L^*(x)$ is the slowly varying function conjugate to $L(x)$ [29]. For every $L(x)$ slowly varying at zero there exists a *conjugate slowly varying function* $L^*(x)$ which is defined such that

$$\lim_{x\to 0}L(x)L^*(xL(x))=1 , \tag{2.12}$$

$$\lim_{x\to 0}L^*(x)L(xL^*(x))=1 , \tag{2.13}$$

$$L^{**}(x)\sim L(x) \quad \text{for } x\to 0 . \tag{2.14}$$

$L^*(x)$ is asymptotically unique in the sense that if there exists another slowly varying function $L'(x)$ with the properties (2.12)–(2.14) then $L'(x)\sim L^*(x)$ for $x\to 0$. Thus to every phase transition of order (λ,Λ) in the pressure there corresponds a conjugate transition in the energy density which is of order (λ^*,Λ^*) with λ^* given by (2.11) and Λ^* related to L^* as Λ to L in (2.9). Phase transitions of order $\lambda=2$ are *self-conjugate* in the sense that $\lambda=\lambda^*$. Phase transitions of order $\lambda=1$ are conjugate to transitions of order $\lambda^*=\infty$ and represent a special limiting situation.

B. Anequilibrium phase transitions

This section turns to the question posed in the Introduction whether phase transitions of order $\lambda<1$ are thermodynamically permissible. Consider $u(s,\rho)$ for a thermodynamic process in which the density $\rho=N/V$ is kept constant and which crosses a critical point at s_c. If the phase transition at s_c is of order $\lambda=\lambda^+=\lambda^-$ then $u(s)$ has the form

$$u(s)=u_{\text{reg}}(s)+u^{\pm}(s)|s-s_c|^{\lambda} , \tag{2.15}$$

where $u_{\text{reg}}(s)$ denotes the regular part and $u^{\pm}(s)$ varies slowly near s_c. Consequently any phase transition with $s_c<\infty$ and order $\lambda<1$ violates the requirement of convexity for u or the condition $u<\infty$ (for $\lambda<0$), and is thus forbidden by the laws of thermodynamics. This appears to restrict thermodynamically admissible transitions to the range $\lambda\geq 1$.

Although the restrictions on the thermodynamic state variables require a finite entropy or energy density, i.e., $s<\infty$ or $u<\infty$, the laws of thermodynamics do not require $s_c<\infty$, i.e., finiteness for the critical point. In fact the simplest solid-fluid phase diagrams in the (s,v) plane are consistent with a critical point at $s_c=\infty$ terminating the solid-fluid coexistence. To exhibit the theoretical possibility of such infinite entropy density transitions it suffices to consider an explicit example which is compatible with the mathematical requirements specified in the previous section. Such an example is given by the following single-valued, continuous, and differentiable energy-density function

$$u(s)=as+b(s^2+c^2)^{1/2} , \tag{2.16}$$

where $a,b,c>0$ and $a>b$. Clearly $T(s)=\partial u/\partial s>0$ and $\partial^2 u/\partial s^2>0$ and thus $u(s)$ is convex and monotonically increasing. $u(s)$ fulfills all requirements for the energy density of a thermodynamically stable system. Note that $u(s)$ exhibits transitions of order $\lambda_u^{\pm}=1$ at $s_c=\pm\infty$. Moreover the thermodynamic system described by Eq. (2.16) has the curious property that the set of possible temperatures is restricted to the range

$$a-b=T_{\min}<T<T_{\max}=a+b . \tag{2.17}$$

The pressure obtained from Eqs. (2.1) and (2.16) reads as

$$p(T)=\{c^2[b^2-(T-a)^2]\}^{1/2} \tag{2.18}$$

and it again exhibits the restricted temperature range. The pressure (2.18) has transitions of order $\lambda_p^{\pm}=\frac{1}{2}$ at $T_{\min}$ and $T_{\max}$, respectively. More generally transitions of order $\lambda_u>0$ in u are related to transitions of order

$$\lambda_p=\frac{\lambda_u}{\lambda_u+1} \tag{2.19}$$

in p [5]. Note that now $0<\lambda_p<1$ while $0<\lambda_u<\infty$.

The simple example (2.16) demonstrates that thermodynamics allows two fundamentally different types of phase transitions: On the one hand traditional phase transitions of order $\lambda_p^{\pm}\geq 1$ and on the other hand unusual phase transitions of order $0<\lambda_p^{\pm}<1$ for which the set of possible equilibrium temperatures appears to be restricted to a subset of the absolute temperature scale. The interest in this observation derives from the fact that equilibrium thermodynamics formally admits transitions whose presence would restrict its own applicability in the sense that the limiting critical temperatures $T_{\min}$ and $T_{\max}$ cannot be reached in any quasistatic thermodynamic process. A quasistatic process is a sequence of state

changes which proceeds infinitely slowly compared to the time scale for the establishment of equilibrium. This raises the question of whether the identification of the absolute temperature scale with the ideal-gas temperature scale remains valid when $\lambda < 1$ transitions are present. In such systems $T_{\min}$ plays the role of absolute zero and $T_{\max}$ that of $T = \infty$. In [5] it was suggested to circumvent the self-limitation to a finite temperature range through multivalued thermodynamic potentials and phase transitions of order $\lambda < 1$ were called nonequilibrium phase transitions because transitions between different sheets cannot occur quasistatically. The present paper, however, restricts all thermodynamic functions to remain single valued. In order to avoid confusion with standard literature usage of the terminus "nonequilibrium phase transitions" I will use instead the word *anequilibrium phase transition* from now on.

The entropy density $s(T) = (\partial p / \partial T)_\mu$ derived from (2.18) diverges to $-\infty$ as $T \to T_{\min} +$. Therefore the third law implies the existence of another special temperature T_0 defined by the condition

$$s(T_0) = 0 \tag{2.20}$$

of vanishing entropy density. Because the third law is of quantum-mechanical origin the temperature T_0 is expected to be the minimal temperature for quantum systems while $T_{\min}$ is the minimal temperature for classical systems. Clearly $T_0 > T_{\min}$ is always fulfilled.

III. STATISTICAL MECHANICS

A. Ensemble limit

Given the thermodynamic classification of phase transitions it is natural to ask whether anequilibrium phase transitions and a statistical-mechanical classification corresponding to the thermodynamic scheme exist for critical behavior in statistical mechanics. These questions are discussed in the following sections. Statistical mechanics for noncritical systems is based on the law of large numbers [30]. This suggests that the theory of critical phenomena may be founded in the theory of stable laws. Although natural this idea is usually rejected because the divergence of correlation lengths and susceptibilities appears to imply that the microscopic random variables are strongly dependent [31–33] while the standard theory of stable laws applies only to weakly dependent or independent variables [28,34,35].

The problem of strong dependence arises from the particular choice of performing the infinite-volume limit and the continuum limit. One usually starts from an infinite-volume lattice theory and then asks for possible continuum (or scaling) limits of the rescaled infinite-volume correlation functions [33]. Depending on whether the rescaled correlation lengths remain finite or not one distinguishes the "massive" and the "massless" scaling limit but in either case the infinite-volume limit has been performed before taking the scaling limit.

The idea of the present paper for basing a statistical classification of critical behavior on the theory of stable laws is related to that of finite-size scaling [36–40] and uses a different method of taking infinite-volume and continuum limits. Consider a d-dimensional simple cubic lattice with lattice spacing $a > 0$ in "block geometry," i.e., having finite side length $L < \infty$ in all d directions. Let X be a scalar observable associated with each lattice point. The lattice represents a discretization of a large but finite statistical-mechanical system [41]. Let the lengths a, L and the parameters Π of the statistical-mechanical system be such that

$$0 < a \ll \xi_X(\Pi) \ll L < \infty \ . \tag{3.1}$$

Thus the system decomposes into a large number of uncorrelated blocks of linear extension ξ_X. The *ensemble limit* is defined as the simultaneous limit in which

$$a \to 0 \ , \quad L \to \infty \ , \quad \Pi \to \Pi_c$$
$$\text{such that } \xi_X(\Pi) \to \xi_X(\Pi_c) < \infty \ . \tag{3.2a}$$

Two cases can be distinguished: In the *critical ensemble limit*

$$0 < \xi_X(\Pi_c) < \infty \ , \tag{3.2b}$$

while for the *noncritical ensemble limit*

$$\xi_X(\Pi_c) = 0 \ . \tag{3.2c}$$

If $N = (L/\xi_X)^d$ denotes the number of uncorrelated blocks of size ξ_X and $M = (\xi_X / a)^d$ is the number of sites in each block then $NM = (L/a)^d$ is the total number of lattice sites. The correlation length ξ_X diverges in units of a in the critical ensemble limit but stays finite in the noncritical ensemble limit. Note also that $N \to \infty$ in the critical ensemble limit while N remains finite in the massive scaling limit or the finite-size scaling limit [37]. The critical ensemble limit generates an infinite ensemble of uncorrelated blocks. This feature allows the application of standard limit theorems for uncorrelated or weakly dependent variables.

Let $X_{iN}(j)$ denote the scalar observable X at lattice site j ($j = 1, \ldots, M$) inside block i ($i = 1, \ldots, N$). Then

$$X_{iN} = \sum_{j=1}^{M} X_{iN}(j) \tag{3.3a}$$

are the *block sums* or *block variables* for block i and

$$X_N = \sum_{i=1}^{N} X_{iN} \tag{3.3b}$$

is the *ensemble sum* or *ensemble variable* for the total system. The $X_{iN}(j)$ are random variables and so are X_{iN} and X_N. Let

$$\tilde{X}_N = (X_N - C_N)/D_N \tag{3.4}$$

denote the normed and centered ensemble sum and let $P_N(x) = \text{Prob}\{X_N < x\}$ be the probability distribution function of X_N. Assuming translation invariance the block variables are uncorrelated and identically distributed. Therefore the limiting distribution of $\tilde{X}_N$ is stable in the critical ensemble limit [28,34,35]. More precisely, if

$$P(x) = \lim_{N \to \infty} P_N(xD_N + C_N) \tag{3.5}$$

denotes the limiting distribution function of the normed ensemble sums (3.4) then the characteristic function $p(k)=\int_{-\infty}^{\infty}\exp(ikx)dP(x)$ of $P(x)$ has the representation

$$\ln p(k)=iCk-D|k|^{\varpi}\left[1-i\zeta\frac{k}{|k|}\omega(k,\varpi)\right], \tag{3.6}$$

where ϖ,ζ,C,D are constants whose ranges are

$$0<\varpi\leq 2\ , \tag{3.7a}$$

$$-1\leq\zeta\leq 1\ , \tag{3.7b}$$

$$-\infty<C<\infty\ , \tag{3.7c}$$

$$D\geq 0\ , \tag{3.7d}$$

and

$$\omega(k,\varpi)=\begin{cases}\tan\left[\dfrac{\varpi\pi}{2}\right], & \text{for } \varpi\neq 1\ ,\\ \dfrac{2}{\pi}\ln|k|\ , & \text{for } \varpi=1\ .\end{cases} \tag{3.8}$$

The constant ϖ is called the *index* of the stable distribution while the parameter ζ characterizes its symmetry or skewness.

If the limit in (3.5) exists and $D>0$ then the norming constants D_N must have the form [28,35]

$$D_N=N^{1/\varpi}\Lambda(N)\ , \tag{3.9}$$

where the function $\Lambda(N)$ is slowly varying at infinity. The case $D>0$ corresponds to the critical ensemble limit, while for $D=0$ the limiting distribution $P(x)$ is degenerate, i.e., concentrated at a single point, corresponding to the noncritical ensemble limit.

The preceding limit theorem implies that in the limit $N\to\infty$ the distribution function of the block sums can be approximated as

$$P_N(x)\approx P\left[\frac{x-C_N}{D_N};\varpi,\zeta,C,D\right], \tag{3.10}$$

where the notation $P(x;\varpi,\zeta,C,D)$ is introduced for stable distributions of index ϖ. The objective in the next section will be to establish a large-N scaling result for $P_N(x)$. To obtain it more information on the common distribution of the individual block variables is required.

The limiting distributions $\tilde{P}(x)$ of the individual block variables X_{iN} are independent of i because of translation invariance and they belong to the domain of attraction of a stable law. The class of possible block variable limits can thus be characterized as follows [35]: In order that the characteristic function $\tilde{p}(k)$ of $\tilde{P}(x)$ belongs to the domain of attraction of a stable law whose characteristic function has the logarithm $-D|k|^{\varpi}[1-i\zeta(k/|k|)\omega(k,\varpi)]$ with ϖ, ζ, D, and $\omega(k,\varpi)$ as in (3.6)–(3.8), it is necessary and sufficient that in the neighborhood of the origin $k=0$

$$\ln\tilde{p}(k)=i\tilde{C}k-D|k|^{\varpi}\tilde{\Lambda}(k)\left[1-i\zeta\frac{k}{|k|}\omega(k,\varpi)\right], \tag{3.11}$$

where $\tilde{C}$ is a constant and $\tilde{\Lambda}(k)$ is a slowly varying function for $k\to 0$.

Equations (3.9) or (3.11) show that each ensemble limit $N,M\to\infty$ is labeled by a set of numbers ϖ,ζ,D with ranges as in (3.7) and a slowly varying function Λ. While D differentiates between critical and noncritical limits ϖ, ζ, and Λ differentiate between different critical ensemble limits. This characterization is reminiscent of the thermodynamic classification scheme and suggests a closer correspondence. To establish such a correspondence it is necessary to relate the generalized orders λ in the thermodynamical classification scheme to the numbers ϖ, ζ occurring in the characterization of ensemble limits. This will be done in the next section.

B. Finite-ensemble scaling

The purpose of the present section is to investigate the N dependence of the probability distribution for ensemble sums X_N in the limit of large N. The scaling relations emerging from this analysis will be called *finite-ensemble scaling* because they are closely related to finite-size-scaling relations by virtue of the similarity between the critical ensemble limit defined above and the finite-size-scaling limit [37]. The question is how to choose the norming and centering constants C_N,D_N in (3.10) given the characterization (3.11) for the individual block variables X_{iN}.

The centering constants C_N in Eq. (3.10) can be eliminated from the problem by setting $C_N=-C'D_N$ where

$$C'=\begin{cases}C\ , & \text{for } \varpi\neq 1\ ,\\ C+\dfrac{2}{\pi}\zeta D\ln D\ , & \text{for } \varpi=1\ ,\end{cases} \tag{3.12}$$

and with this choice (3.10) becomes

$$P_N(x)\approx P\left[\frac{x-C_N}{D_N};\varpi,\zeta,C,D\right]$$

$$=P\left[\frac{x}{D^{1/\varpi}D_N};\varpi,\zeta,0,1\right]. \tag{3.13}$$

Although the general form of D_N is known from (3.9) it remains to establish the relationship between the slowly varying functions in (3.9) and (3.11). Once this relation is established Eq. (3.13) represents a finite N scaling formula for a system in which the individual block variable limits are characterized by (3.11).

The limiting distribution functions of the individual block variables X_{iN} have characteristic functions as given by (3.11). Introduce

$$R(k)=|k|^{\varpi}\mathfrak{L}(k^{\varpi})\ , \tag{3.14}$$

where the slowly varying function $\mathfrak{L}(x)$ is defined through

$$\tilde{\Lambda}(k)=\mathfrak{L}(k^{\varpi}) \tag{3.15}$$

and $\tilde{\Lambda}(k)$ is the slowly varying function appearing in (3.11). For sufficiently large N the norming constants D_N

are chosen as

$$D_N^{-1}=\inf\left\{k>0:R(k)=\frac{D}{N}\right\} \tag{3.16}$$

which is possible because $R(k)\to 0$ for $k\to 0$ and $R(k)$ is continuous in a neighborhood of zero. Then for small k [35]

$$\lim_{N\to\infty}\left\{\tilde{p}\left[\frac{k}{D_N}\right]\right\}^N=\lim_{N\to\infty}\exp\left\{-NR\left[\frac{1}{D_N}\right]\frac{R\left[\frac{k}{D_N}\right]}{R\left[\frac{1}{D_N}\right]}\left[1+i\zeta\frac{k}{|k|}\omega(k,\varpi)\right]\right\}$$
$$=\exp\left\{-D|k|^{\varpi}\left[1+i\zeta\frac{k}{|k|}\omega(k,\varpi)\right]\right\}. \tag{3.17}$$

It follows that $D\approx NR(1/D_N)$ for sufficiently large N and this determines D_N in terms of $\tilde{\Lambda}(k)$ and ϖ as

$$D_N=\left[\frac{N}{D\mathfrak{L}^*(N^{-1})}\right]^{1/\varpi}, \tag{3.18}$$

where $\mathfrak{L}^*(x)$ is the conjugate slowly varying function to $\mathfrak{L}(x)$ defined in Eq. (3.15). The slowly varying function $\Lambda(N)$ appearing in (3.9) is thus given as

$$\Lambda(N)=\left[D\mathfrak{L}^*\left[\frac{1}{N}\right]\right]^{-1/\varpi}. \tag{3.19}$$

in terms of $\tilde{\Lambda}(k)$ appearing in the limiting distributions (3.11) for the individual blocks.

Finally Eq. (3.13) gives the finite-ensemble scaling for the distribution of ensemble variables X_N in the large-N limit

$$P_N(x)=P\left[\frac{x\mathfrak{L}^{*1/\varpi}(N^{-1})}{N^{1/\varpi}};\varpi,\zeta,0,1\right]. \tag{3.20}$$

More interesting than the ensemble variables X_N are the *ensemble averages* defined as $\bar{X}_N=X_N/(NM)$. The probability distribution function $\bar{P}_N(\bar{x})$ for the ensemble averages $\bar{X}_N$ has the finite-ensemble-scaling form

$$\bar{P}_N(\bar{x})=\bar{P}\left[\frac{\bar{x}\mathfrak{L}^{*1/\varpi}(N^{-1})}{N^{(1-\varpi)/\varpi}};\varpi,\zeta,0,1\right]. \tag{3.21}$$

If $N=(L/\xi_X)^d$ is expressed in terms of the system size L they are seen to be closely related to finite-size-scaling theory. Note that Eqs. (3.20) and (3.21) are derived without reference to a particular model or approximate critical Hamiltonian such as the Landau-Ginzburg-Wilson Hamiltonian. They are generally valid for all translation-invariant critical systems, i.e., systems for which the basic limit distribution (3.5) is not degenerate.

C. Identification of exponents and statistical classification scheme

It is now possible to consider the correspondence between the statistical classification in terms of ϖ, ζ, and $\tilde{\Lambda}$ and the thermodynamic classification in terms of λ and Λ. To do this the index ϖ in (3.20) and (3.21) must be related to the critical exponents. This is immediately possible from (3.21) by considering for example the order parameter Ψ. Setting $X=\Psi$, taking the derivative with respect to $\bar{x}$ in Eq. (3.21) and using $N=(L/\xi_\Psi)^d$ one finds that the kth moment $\langle\bar{\Psi}^k\rangle$ of the order parameter scales with system size as $\langle\bar{\Psi}^k\rangle\sim L^{kd(\varpi-1)/\varpi}$. Comparing to standard finite-size-scaling theory [36–40] relates ϖ to the thermodynamic exponents as

$$\varpi_\Psi=\frac{\gamma+2\beta}{\gamma+\beta}=1+\frac{1}{\delta}=\lambda_\Psi, \tag{3.22}$$

where β and γ are the order parameter and susceptibility exponents, and λ_Ψ was defined in (2.5). Similarly for the energy density $X=\mathcal{E}$ one finds

$$\varpi_{\mathcal{E}}=2-\alpha=\lambda_{\mathcal{E}}, \tag{3.23}$$

where $\lambda_{\mathcal{E}}$ is the thermal order of (2.5). This suggests that the correspondence between the statistical and the thermodynamic classification of phase transitions is given generally as $\varpi=\lambda$. Note that second-order (i.e., self-conjugate) phase transition occupy again a special place in the statistical classification scheme because of the bound $\varpi\le 2$ in (3.7a). This fact will be related below to violations of hyperscaling relations.

Anequilibrium phase transitions with order $\lambda<1$ correspond to stable limit distributions with index $\varpi<1$. Thus anequilibrium transitions are not only predicted by equilibrium thermodynamics but also by equilibrium statistical mechanics. The fact that anequilibrium transitions restrict the range of equilibrium temperatures as in (2.17) is mirrored by the fact that expectation values of averages diverge in the critical ensemble limit for anequilibrium critical points with $\varpi<1$. This implies that the traditional formulation of statistical mechanics becomes inapplicable at anequilibrium critical points just as traditional thermodynamics becomes inapplicable.

While the general correspondence between $\lambda<1$ and $\varpi<1$ is reassuring it is not sufficient to establish the existence of anequilibrium phase transitions in statistical mechanics. To demonstrate their existence requires a possibly exact calculation of the partition sum for a concrete statistical-mechanical model. It is possible to demonstrate the existence of anequilibrium transitions in

this way. A concrete example occurs in what is perhaps the simplest model in the theory of critical phenomena, namely, the one-dimensional Gaussian model [42]. This finding is important because the Gaussian model is of central importance in the modern theory of critical phenomena as the starting point for systematic perturbative calculations [6]. The model Hamiltonian is $\mathcal{H}=-(J/2)\sum\Psi_i\Psi_j$ where the sum runs over all nearest-neighbour pairs of lattice sites i,j and the continuous spin variables Ψ_i have a Gaussian single spin measure proportional to $\exp[-\sigma\Psi_i^2]$. The limiting free-energy density for the one-dimensional Gaussian model is well known and it reads

$$-\frac{f(T)}{k_BT}=\frac{1}{2}\ln\pi-\frac{1}{2}\ln\left[\frac{1}{2}[\sigma+(\sigma^2-K^2)^{1/2}]\right], \quad (3.24)$$

where $K=J/(k_BT)$ and k_B denotes Boltzmann's constant. The exact free-energy density (3.24) for the one-dimensional Gaussian model exhibits an anequilibrium transition of order $\lambda_{\mathcal{E}}=\frac{1}{2}$ at the critical temperature $T_{\min}=J/(k_B\sigma)$.

D. General mechanism for the violation of hyperscaling

The identification $\lambda=\varpi$ cannot hold for all values of $\lambda>0$ because $\varpi\le 2$ is required by (3.7). The new restriction $\varpi\le 2$ is now seen to be related to the violation of hyperscaling and the breakdown of finite-size scaling for thermal fluctuations in dimensions $d>4$. Consider the class of statistical-mechanical models obeying the Lebowitz inequality for the four-point functions and infrared bounds for the two-point functions [33]. For such models the susceptibility exponent γ obeys $\gamma\ge 1$ and the correlation-function exponent η obeys $\eta\ge 0$. Then using $\varpi_{\mathcal{E}}\le 2$, the Fisher inequality $\gamma\le(2-\eta)\nu$, the hyperscaling relation $d\nu=2-\alpha$, and relation (3.23) the following chain of inequalities is obtained:

$$d\le d\gamma\le(2-\eta)d\nu=(2-\eta)(2-\alpha)$$
$$=(2-\eta)\varpi_{\mathcal{E}}\le 2(2-\eta)\le 4 . \quad (3.25)$$

For general models hyperscaling may fail at $\varpi=2$ because there are distributions with nonalgebraic tails within the domain of attraction of the normal law. Note that in this way the inequality $\varpi\le 2$ provides a general mechanism for the breakdown of hyperscaling independent of identifying dangerous irrelevant variables in a particular model. Analogous breakdown phenomena are expected to occur for critical fluctuations in observables other than the energy density.

IV. TEMPERATURE RENORMALIZATION

The presence of anequilibrium transitions in a statistical-mechanical system $\mathfrak{S}$ implies strong fluctuations near the transition point. In fact at the transition point the fluctuations become so strong that a canonical or thermodynamical description of the system becomes impossible because the ensemble-averaged energy or entropy diverges in the infinite system. The underlying microscopic dynamics, however, remains well defined in terms of a classical or quantum-mechanical microscopic Hamiltonian. The total energy of the system remains defined and conserved and the system can be described microcanonically. If the system $\mathfrak{S}$ undergoes anequilibrium transitions at $T^{\mathfrak{S}}_{\min}$ and $T^{\mathfrak{S}}_{\max}$ then heating or cooling across $T^{\mathfrak{S}}_{\min}$ or $T^{\mathfrak{S}}_{\max}$ cannot occur quasistatically and the system must fall out of equilibrium when attempting it.

Consider now the usual setup for the canonical ensemble in which $\mathfrak{S}$ is weakly coupled to a reservoir $\mathfrak{R}$. What happens if the reservoir itself undergoes anequilibrium transitions at $T^{\mathfrak{R}}_{\min}$ and $T^{\mathfrak{R}}_{\max}$? Clearly, the combined system $\mathfrak{R}\cup\mathfrak{S}$ has a Hamiltonian description and can always be treated in the microcanonical ensemble. But what happens to a canonical description? One expects that the canonical description should remain applicable as long as $T^{\mathfrak{R}}_{\min}$ is very small and $T^{\mathfrak{R}}_{\max}$ very large, i.e., for $T^{\mathfrak{R}}_{\min}/T\ll 1\ll T^{\mathfrak{R}}_{\max}/T$, while the temperature dependence of the results should become modified otherwise. The temperature dependence of the results of canonical calculations enters through the Lagrange parameter $\beta(T)$ for the average energy. The Lagrange parameter β appearing in the canonical and the grand canonical ensembles is a universal (i.e., $\mathfrak{S}$ independent) function of absolute temperature T and at the same time a property of the reservoir $\mathfrak{R}$. In fact if the reservoir $\mathfrak{R}$ consists of a large number of weakly interacting subsystems (e.g., particles) then β^{-1} is proportional to the limiting energy per subsystem of $\mathfrak{R}$ [30]. Because β is related to the energy density of the reservoir unusual temperature dependence must be expected whenever the reservoir substance itself shows anequilibrium transitions. If the reservoir is described quantum mechanically then $T^{\mathfrak{R}}_0$ defined by Eq. (2.20) will be the lowest temperature of the reservoir corresponding to the ground state of the reservoir Hamiltonian. For classical reservoirs $T^{\mathfrak{R}}_{\min}$ will be the lowest temperature. If the reservoir has no anequilibrium transitions, i.e., if $T^{\mathfrak{R}}_0=0$ and $T^{\mathfrak{R}}_{\max}=\infty$ then the temperature dependence must have the usual universal form,

$$\beta(T)=\frac{1}{k_BT} . \quad (4.1)$$

If, however, $T^{\mathfrak{R}}_0>0$ and $T^{\mathfrak{R}}_{\max}<\infty$ the form (4.1) cannot be correct. The energy per subsystem of the reservoir is proportional to β^{-1} and it diverges as T approaches $T^{\mathfrak{R}}_{\max}$. Thus for finite $T^{\mathfrak{R}}_{\max}<\infty$ one must have

$$\beta(T^{\mathfrak{R}}_{\max})=0 \quad (4.2)$$

and this contradicts (4.1) because $1/(k_BT^{\mathfrak{R}}_{\max})>0$. Similarly

$$\beta^{-1}(T^{\mathfrak{R}}_0)=0 \quad (4.3)$$

also violates (4.1) because $T^{\mathfrak{R}}_0>T^{\mathfrak{R}}_{\min}>0$. To satisfy the relations (4.2) and (4.3) the temperature T and the parameter β must in general become renormalized into

$$'T=T-T^{\mathfrak{R}}_0 , \quad (4.4)$$

$$'\beta=\beta-\beta(T^{\mathfrak{R}}_{\max}) , \quad (4.5)$$

whenever the reservoir undergoes anequilibrium transitions. This suggests that canonical averages $\langle\mathcal{O}\rangle$ of an

observable $\mathcal{O}$ will in general depend on temperature through (4.4) and (4.5). For classical reservoirs $T_0^{\Re}$ in (4.4) has to be replaced by $T_{\min}^{\Re}$. Equations (4.4) and (4.5) are general predictions for the temperature dependence of canonical averages in systems with anequilibrium transitions resulting from the requirement of consistency for the interpretation of β. These general results are corroborated by the explicit solution (3.24) for the Gaussian model.

The main result expressed in (4.4) and (4.5) is the fact that the temperature dependence of canonical averages depends on the nature of the reservoir with which the system is equilibrated whenever the reservoir undergoes phase transitions of order less than 1. Note that in every theoretical evaluation of a canonical partition sum it is implicitly assumed that the system can be equilibrated with a reservoir $\Re$ such as an ideal gas with $T_0^{\Re}=0$ and $T_{\max}^{\Re}=\infty$ not undergoing anequilibrium transitions. This implicit assumption need not be fulfilled in experiment. In fact experimentally the reservoir is very often of the same material as the system itself because the system under study is part of a much larger (ideally infinite) sample. In that case $T_0^{\Re}=T_0^{\mathfrak{S}}$ and $T_{\max}^{\Re}=T_{\max}^{\mathfrak{S}}$ and the temperature renormalization (4.4) must be expected to become relevant as T approaches $T_0^{\mathfrak{S}}$.

V. ANEQUILIBRIUM DYNAMICS

The divergence of expectation values for block energies makes it clear that the concept of stationarity for macroscopic block variables requires modification whenever the system is at an anequilibrium transition. In the ensemble limit each block becomes an infinite system. The present section discusses general aspects of the macroscopic dynamics.

The connection between the static properties discussed in the preceding sections and the macroscopic time evolution is provided by the ergodic hypothesis. The ergodic hypothesis allows one to view a sequence of block variables X_{iN} as a sequence of snapshots $X_N(t_i)=X_{iN}$ representing a possible stroboscopically recorded time evolution of a single block. This temporal embedding of an arbitrary sequence of block variables defines the sequence t_i of time points as a strictly increasing stochastic sequence with stationary and positive random increments $\tau_i=t_{i-1}-t_i>0$. The time instants $t_i>0$ are positive random variables and, assuming $t_0=0$, one has $t_N=\tau_1+\cdots+\tau_N$. The critical ensemble limit corresponds now to the long-time limit of a large system. In this limit the distribution function $P_N(t)$ of the variables t_N converges to that of a *one-sided* stable law with index ϖ_t and symmetry parameter ζ_t. In general the index ϖ_t will be a new exponent which is different from the static exponents and also depends on the macroscopic observable X of interest.

An important observation for the temporal process defined through a sequence X_{iN} of block variables is that the limiting law must be one sided, i.e., the limiting distribution function $P(t)$ is nonzero only for $t>0$. This places the restrictions $\zeta_t=1$ and $0<\varpi_t\leq 1$ on the parameters of the possible limiting distributions. A second important observation is that each distribution $P_N(t)$ defines a stable convolution semigroup $\mathfrak{P}(D_N)$ [28] with a continuous N-dependent parameter D_N given in (3.18). The convolution operators defined as

$$\mathfrak{P}(D_N)f(t)=\int_{-\infty}^{\infty}f(t-t')dP_N(t') \tag{5.1}$$

with $D_N=[N/D\Omega^*(N^{-1})]^{1/\varpi_t}$, $D>0$ have the semigroup property [28]

$$\mathfrak{P}(D_N)\mathfrak{P}(D_N')=\mathfrak{P}(D_N+D_N')\ , \tag{5.2}$$

and this semigroup has a generator $\mathfrak{A}$ defined by $\mathfrak{A}=\lim_{D_N\to 0^+}D_N^{-1}[\mathfrak{P}(D_N)-\mathfrak{I}]$, where $\mathfrak{I}$ denotes the identity operator [28]. For sufficiently large N, $P_N(t)$ can be replaced by a one-sided stable distribution. The ergodic hypothesis allows to identify the generator of the semigroup (5.2) with the generator for the time transformation of the macroscopic observable $X(t)$. The generators $\mathfrak{A}_{\varpi_t}$ of one-sided stable semigroups with index ϖ_t are well known [28] to be proportional to the fractional time derivatives $d^{\varpi_t}/dt^{\varpi_t}$. The case $\varpi_t=1$ is of special importance because in this case the distribution $P_N(t)$ is degenerate, the semigroup is a semigroup of translations and the corresponding generator is the usual derivative d/dt. The preceding observations imply that the definition of stationarity for the time dependence of a macroscopic block variable $X(t)$ must be generalized into

$$\frac{d^{\varpi_t}}{dt^{\varpi_t}}X(t)=0 \tag{5.3}$$

which reduces to the traditional definition $(d/dt)X(t)=0$ only for the special limiting case $\varpi_t=1$. It is important to note that (5.3) holds only for *macroscopic block variables but not for microscopic site variables.* The main result is that fractional linear differential operators appear naturally as the generators of time transformations for the observables of infinite systems by virtue of the ensemble limit combined with the ergodic hypothesis. The solution to (5.3) will in general be time dependent according to

$$X(t)=C_0t^{\varpi_t-1}\ , \tag{5.4}$$

where C_0 is a constant. Only for $\varpi_t=1$ does (5.4) reduce to constant $X(t)$. In general stationarity is already reached when $X(t)$ decays algebraically. The main consequence of the generalized concept of stationarity expressed in (5.3) is that algebraic time decays of *macroscopic* observables may in fact be stationary.

A further consequence of fractional time derivatives as generators of the time evolution for macroscopic observables is that the equations of motion for a macroscopic observable $X(t)$ become generalized into

$$\frac{d^{\varpi_t}}{dt^{\varpi_t}}X(t)=i\mathcal{L}(t)X(t)\ , \tag{5.5}$$

where $\mathcal{L}$ denotes a generalized Liouville operator of the system which may be explicitly time dependent. For $\varpi_t=1$ and $\mathcal{L}(t)X=(i/\hbar)[\mathcal{H}(t),X]$ reduces to the equations of motion in the Heisenberg representation for a

system with a time-dependent Hamiltonian $\mathcal{H}(t)$.

Equation (5.5) has an interesting formal solution. Laplace transformation yields the expression

$$X(u)=u^{\varpi_t-1}(u^{\varpi_t}-i\mathcal{L})^{-1}X_0 , \tag{5.6}$$

where $X_0=X(t=0)$, u denotes the Laplacian spectral variable, and $\mathcal{L}$ has been assumed to be time independent. Mellin transformation $\mathfrak{M}\{f\}(s)=\int_0^\infty f(t)t^{s-1}dt$ gives the formal result

$$X(s)=\frac{1}{\varpi_t}\frac{\Gamma(s/\varpi_t)\Gamma(1-(s/\varpi_t))}{\Gamma(1-s)}(i\mathcal{L}^{-1})^{s/\varpi_t}X_0 , \tag{5.7}$$

where the relation $\Gamma(1-s)\mathfrak{M}\{f\}(s)=\mathfrak{M}\{\mathfrak{L}\{f\}(u)\}(1-s)$ between Mellin and Laplace transforms has been used. Comparing the inverse Mellin transform with the definition of the H function given in the Appendix yields

$$X(t)=\frac{1}{\varpi_t}H_{1,2}^{1,1}\left[(-i\mathcal{L})^{1/\varpi_t}t\,\middle|\,\begin{matrix}\left(0,\frac{1}{\varpi_t}\right)\\ \left(0,\frac{1}{\varpi_t}\right)\end{matrix}\;\begin{matrix}\\ (0,1)\end{matrix}\right]X_0 . \tag{5.8}$$

This result may be rewritten more conveniently by exploiting the fact that the H function $H_{1,2}^{1,1}$ is closely related to the class of Mittag-Leffler functions. In this way the series representation

$$X(t)=\left[\sum_{k=0}^{\infty}\frac{t^{k\varpi_t}}{\Gamma(k\omega_t+1)}(i\mathcal{L})^k\right]X_0 \tag{5.9}$$

for the result (5.8) is obtained. The result (5.9) describes the generalized solution for nonstationary macroscopic observables whose macroscopic time evolution is governed by $\mathcal{L}$ and for $\varpi_t=1$ this is seen to reduce to the familiar result $X(t)=e^{i\mathcal{L}t}X_0$. Evidently the solution (5.9) represents very slow nonexponential dynamics approaching algebraic time decay in the long-time limit.

VI. CONCLUSION

The present paper has discussed the existence and properties of phase transitions of order less than unity which were termed anequilibrium transitions. The following results of direct physical relevance were obtained: (i) Anequilibrium transitions are allowed by the laws of thermodynamics and they occur in models of statistical mechanics. (ii) The existence of anequilibrium transitions implies the existence of "nonequilibrium temperatures" at which the system cannot be described by equilibrium statistical physics. This result points towards a possible incompleteness in the foundations of statistical physics. (iii) The paper has presented a general derivation of finite-size scaling without use of renormalization-group theory. (iv) A mechanism for the breakdown of hyperscaling was found which does not invoke dangerous irrelevant variables. (v) Anequilibrium transitions exhibit an entropy catastrophy and are asymmetric. They require a renormalization of temperature if the reservoir in the canonical ensemble is made of the same substance as the system itself. The general classification theory predicts modified generators for the time evolution of macroscopic observables in systems with $\varpi_t<1$. The latter systems exhibit algebraically decaying stationary states and nonexponential relaxation given by H functions. Given these results it seems not too far fetched to suggest that anequilibrium transitions are promising candidates for the elusive glass transition although much more experimental and theoretical work is required to establish this point.

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APPENDIX: DEFINITION OF H FUNCTIONS

The H function is defined as [43]

$$H_{P,Q}^{m,n}\left[z\,\middle|\,\begin{matrix}(\alpha_1,A_1) & \cdots & (\alpha_P,A_P)\\ (\beta_1,B_1) & \cdots & (\beta_Q,B_Q)\end{matrix}\right]=\frac{1}{2\pi i}\int_{\mathfrak{C}}\frac{\prod_{j=1}^{m}\Gamma(\beta_j-B_js)\prod_{j=1}^{n}\Gamma(1-\alpha_j+A_js)}{\prod_{j=m+1}^{Q}\Gamma(1-\beta_j+B_js)\prod_{j=n+1}^{P}\Gamma(\alpha_j-A_js)}z^{-s}ds ,$$

where $\mathfrak{C}$ is a contour from $c-i\infty$ to $c+i\infty$ separating the poles of $\Gamma(\beta_j-B_js)$, $j=1,\ldots,m$ from those of $\Gamma(1-\alpha_j+A_js)$, $j=1,\ldots,n$. Empty products are interpreted as unity. The integers m,n,P,Q satisfy $0\le m\le Q$ and $0\le n\le P$. The coefficients A_j and B_j are positive real numbers and the complex parameters α_j,β_j are such that no poles in the integrand coincide. If

$$0<\sum_{j=1}^{n}A_j-\sum_{j=n+1}^{P}A_j+\sum_{j=1}^{m}B_j-\sum_{j=m+1}^{Q}B_j=\Omega$$

then integral converges absolutely and defines the H func-

tion in the sector $|\arg(z)| < \frac{1}{2}\Omega\pi$. The H function is also well defined when either

$$\delta = \sum_{j=1}^{Q} B_j - \sum_{j=1}^{P} A_j > 0 \quad \text{and} \quad 0 < |z| < \infty$$

or

$$\delta = 0 \quad \text{and} \quad 0 < |z| < R \equiv \prod_{j=1}^{P} A_j^{-A_j} \prod_{j=1}^{Q} B_j^{B_j} .$$

The H function is a generalization of Meijer's G function and contains many of the known special functions. In particular Mittag-Leffler and generalized Mittag-Leffler functions are special cases of the H function.

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